Exercise 1.5.9

Determine the equilibrium temperature distribution inside a circular annulus ($r_1 \leq r \leq r_2$):

(a) if the outer radius is at temperature T_2 and the inner at T_1

(b) if the outer radius is insulated and the inner radius is at temperature T_1

Solution

The governing equation for the temperature in the annulus, assuming radial symmetry, is

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad r_1 \leq r \leq r_2.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of r now.

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) \quad \rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

To solve the differential equation, multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{du}{dr} = C_1$$

Divide both sides by r.

$$\frac{du}{dr} = \frac{C_1}{r}$$

Part (a)

Integrate both sides with respect to r once more.

$$u(r) = C_1 \ln r + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$u(r_1) = C_1 \ln r_1 + C_2 = T_1 \tag{1}$$
$$u(r_2) = C_1 \ln r_2 + C_2 = T_2 \tag{2}$$

Subtract equation (2) from equation (1) to get an equation for C_1.

$$C_1(\ln r_1 - \ln r_2) = T_1 - T_2$$
$$C_1 \ln\frac{r_1}{r_2} = T_1 - T_2 \quad \rightarrow \quad C_1 = \frac{T_1 - T_2}{\ln\frac{r_1}{r_2}}$$

Solve equation (1) for C_2.

$$\begin{aligned} C_2 &= T_1 - C_1 \ln r_1 \\ &= T_1 - \frac{T_1 - T_2}{\ln\frac{r_1}{r_2}} \ln r_1 \end{aligned}$$

The equilibrium temperature distribution is then

$$\begin{aligned}
u(r) &= C_1 \ln r + C_2 \\
&= \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r + T_1 - \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln r_1 \\
&= T_1 + \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} (\ln r - \ln r_1) \\
&= T_1 + \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_1} \\
&= T_1 + \frac{T_2 - T_1}{\ln \frac{r_2}{r_1}} \ln \frac{r}{r_1}
\end{aligned}$$

Therefore, the equilibrium temperature distribution is

$$u(r) = T_1 + (T_2 - T_1) \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}.$$

Part (b)

If the outer radius is insulated, then du/dr must be zero there.

$$\frac{du}{dr}(r_2) = \frac{C_3}{r_2} = 0 \quad \rightarrow \quad C_3 = 0$$

The differential equation becomes

$$\frac{du}{dr} = 0.$$

Integrate both sides with respect to r once more.

$$u(r) = C_4$$

Apply the second boundary condition to determine C_4.

$$u(r_1) = C_4 = T_1$$

Therefore, the equilibrium temperature distribution is

$$u(r) = T_1.$$